Analysis and Design of Analog Integrated Circuits
Lecture 14

Noise Spectral Analysis for Circuit Elements

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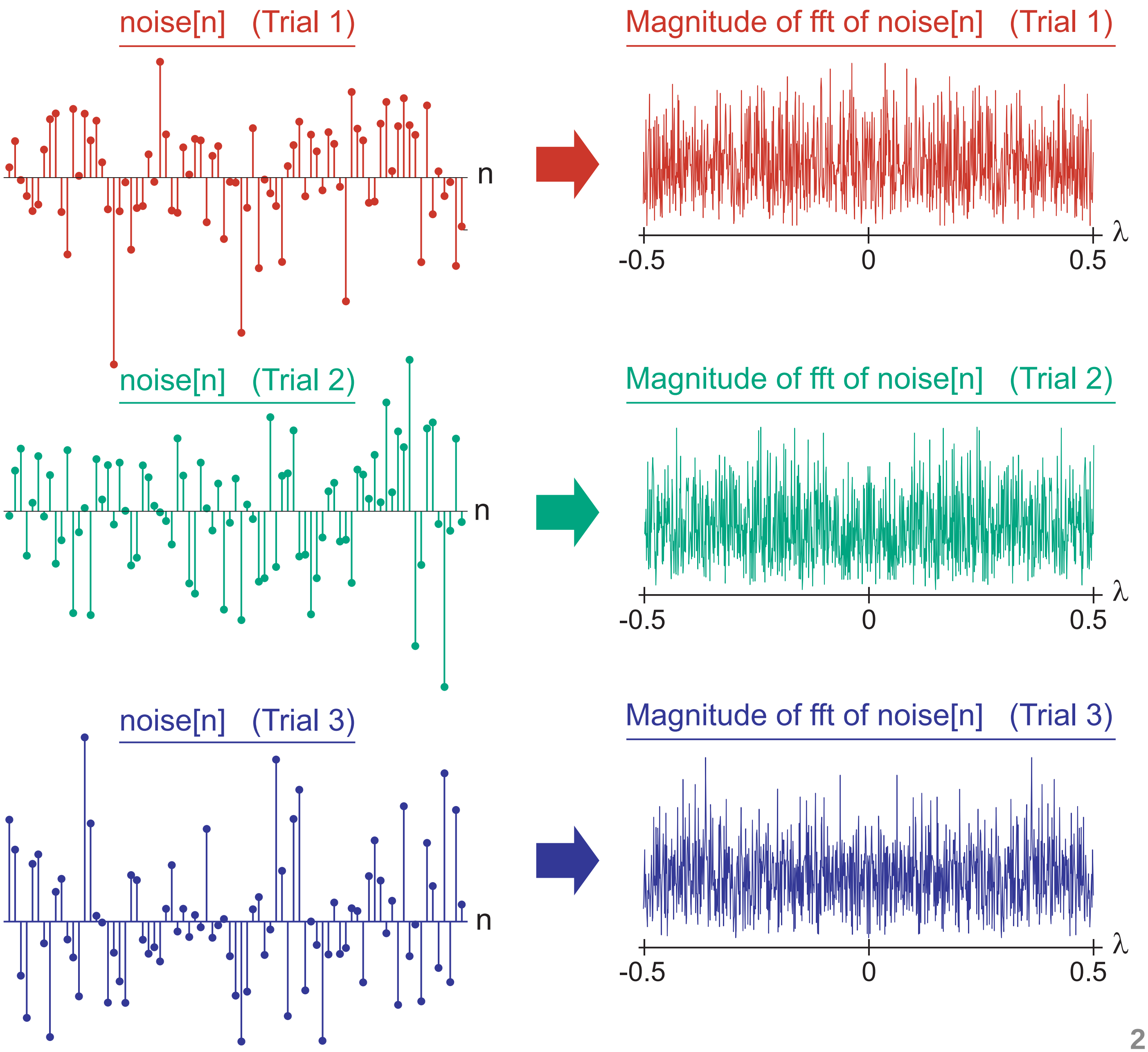
Recall Frequency Domain View of Random Process

- **It is valid to take the *FFT* of a sequence from a given trial**
- **However, notice that the *FFT* result changes across trials**
  - **Fourier Transform of a random process is undefined !**
  - **We need a new tool called spectral analysis**



Expectation of a Random Variable

- **The *expectation* of random variable *y* is defined as**

$$E(y) = \int_{-\infty}^{\infty} y f_y(y) dy$$

  - **We see that:**

$$E(y) = \int_{-\infty}^{\infty} y f_y(y) dy = \mu_y$$

$$E((y-\mu_y)^2) = \int_{-\infty}^{\infty} (y-\mu_y)^2 f_y(y) dy = \sigma_y^2$$

  - **In the case where $\mu_y = 0$ (i.e., the mean of *y* is 0)**

$$E(y^2) = E((y-\mu_y)^2) = \sigma_y^2$$

    - $E(y^2)$ is called the second moment of random variable *y*

Independence of Random Variables

- **Consider two random variables *x* and *y***
  - ***x* and *y* are said to be independent if and only if**

$$f(x,y) = f(x)f(y)$$

    - Where *f(x,y)* is the joint probability distribution of *x* and *y*
  - **which implies**

$$E(xy) = \int_{-\infty}^{\infty} xyf(x,y)dxdy = \int_{-\infty}^{\infty} xf(x)dx \int_{-\infty}^{\infty} yf(x)dy$$

$$\Rightarrow \quad E(xy) = E(x)E(y)$$

    - The above relationship is also true under a less strict condition called *linear independence*
    - If *x* and *y* are zero mean, then *E(xy)* = 0 implies that *x* and *y* are *uncorrelated*

Autocorrelation and Spectral Density (Discrete-Time)

- **Assume a zero mean, stationary random process *x[n]*:**
  - **The autocorrelation of *x[n]* is defined as:**

$$R_{xx}[m] = E(x[n] \cdot x[n+m])$$

    - Note that:

$$R_{xx}[0] = E(x^2[n]) = \sigma_x^2$$

  - **The power spectral density of random process *x[n]* is defined as**

$$S_x(\lambda) = \sum_{m=-\infty}^{\infty} R_{xx}[m] e^{-j2\pi\lambda m}$$

    - Note that $\lambda = fT$, where *f* is frequency (in Hz) and *T* is the sample period of the process (in units of seconds)
    - Power spectral density of *x[n]* is essentially the (Discrete-Time) Fourier Transform of the autocorrelation of *x[n]*

Implications of Independence (Discrete-Time)

- If the samples of a zero mean random process, *x[n]*, are independent of each other, this implies

$$R_{xx}[m] = E(x[n]x[n+m])$$

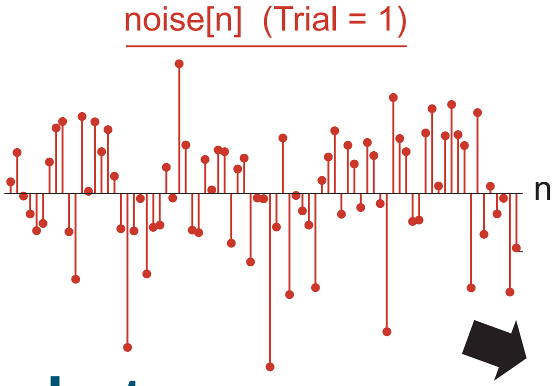
$$= \begin{cases} E(x^2[n]) = \sigma_x^2, & m = 0 \\ E(x[n])E(x[n+m]) = 0, & m \neq 0 \end{cases}$$

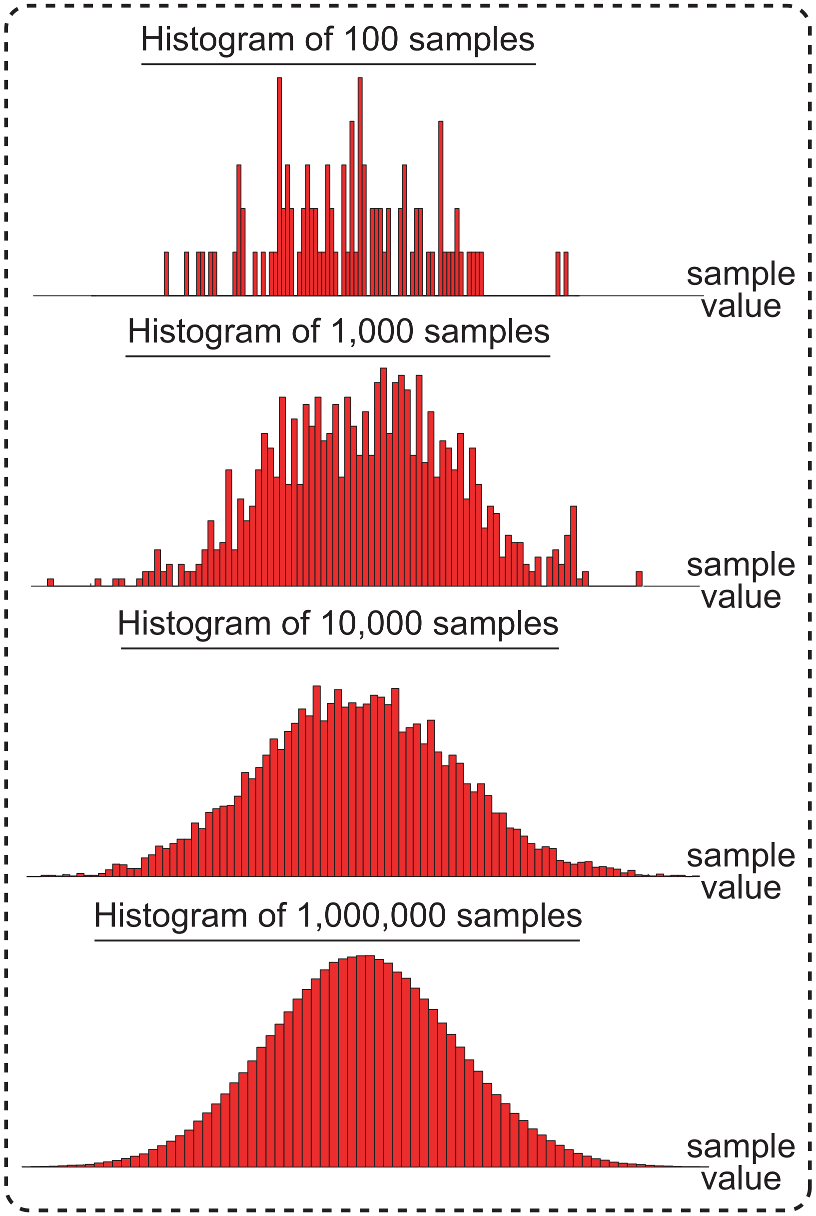
- The corresponding power spectral density is then calculated as

$$\Rightarrow \; S_x(\lambda) = \sum_{m=-\infty}^{\infty} R_{xx}[m]e^{-j2\pi\lambda m} = \sigma_x^2$$

  - This is a known as a *white* random process, whose spectral density is flat across all frequencies

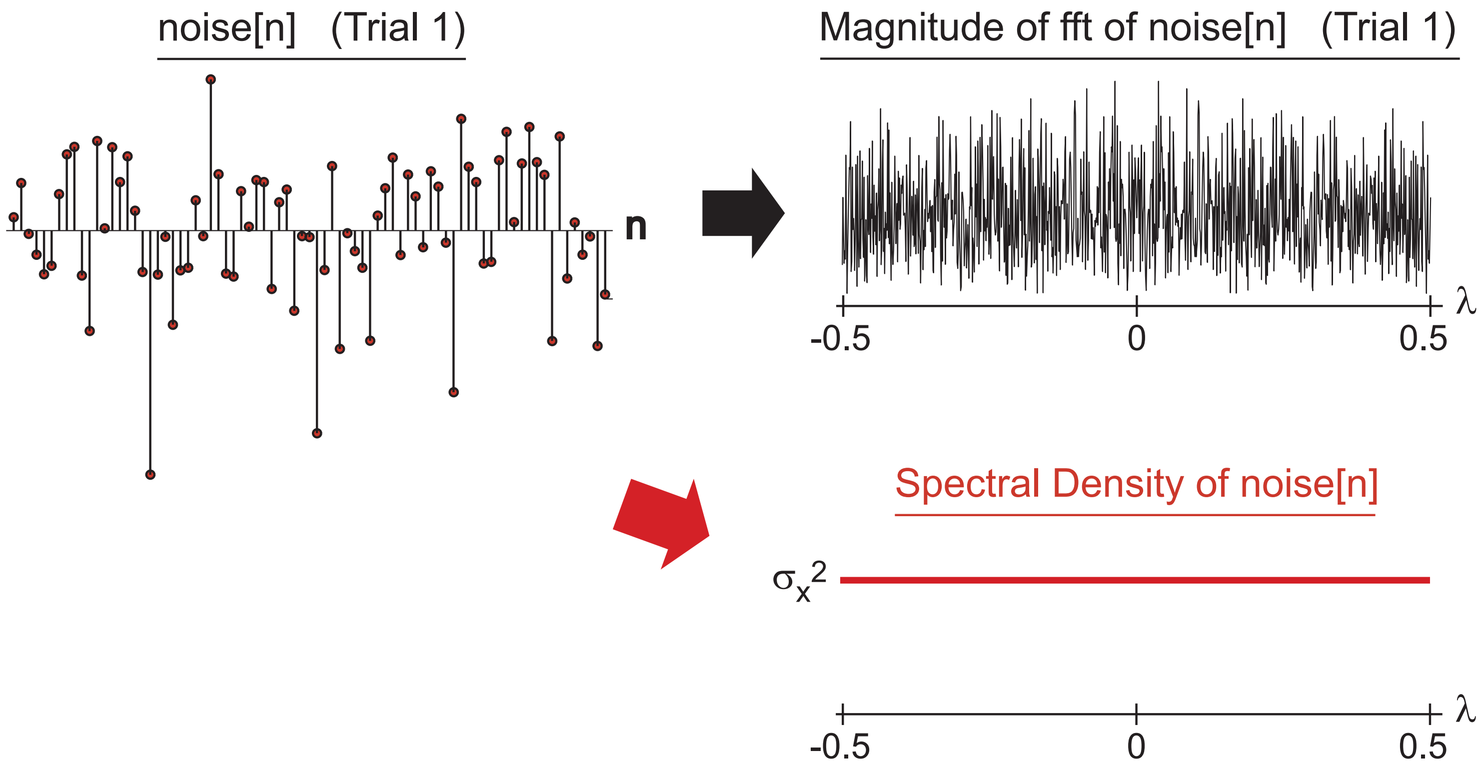
Understanding White Random Processes





- **Independence between samples implies that previous samples provide no benefit in trying to predict the value of the current sample**
- **For Gaussian white processes, the best we can do is use the Gaussian PDF to determine the probability of a sample being within a given range**
  - **Variance of the process is a key parameter**

Spectral Density of a White Process (Discrete-Time)



- **The spectral density of a white process is well defined**
  - **This is in contrast to the FFT of a white process, which varies between different trials of the process**
  - **Note that the spectral density is *double-sided* since it is based on the Fourier Transform (which is defined for both positive and negative frequencies)**

Autocorrelation and Spectral Density (Continuous-Time)

- **Assume a zero mean, stationary random process *x(t)*:**
  - **The autocorrelation of *x(t)* is defined as:**

$$R_{xx}(\tau) = E(x(t) \cdot x(t+\tau))$$

    - Note that

$$R_{xx}(0) = E(x^2(t)) = \sigma_x^2$$

  - **The power spectral density of random process *x(t)* is defined as**

$$S_x(f) = \int_{\tau=-\infty}^{\infty} R_{xx}(\tau) e^{-j2\pi f\tau} d\tau$$

    - Again, the power spectral density corresponds to the Fourier Transform of the autocorrelation function of the random process *x(t)*

White Random Process (Continuous-Time)

- **Assume a zero mean, stationary random process *x(t)*:**
  - **Assuming that the samples of a random process, *x(t)*, are independent of each other, this implies**

$$R_{xx}(\tau) = E(x(t)x(t+\tau)) = N_o\delta(t)$$

    - Where δ(t) is known as the delta function with properties:

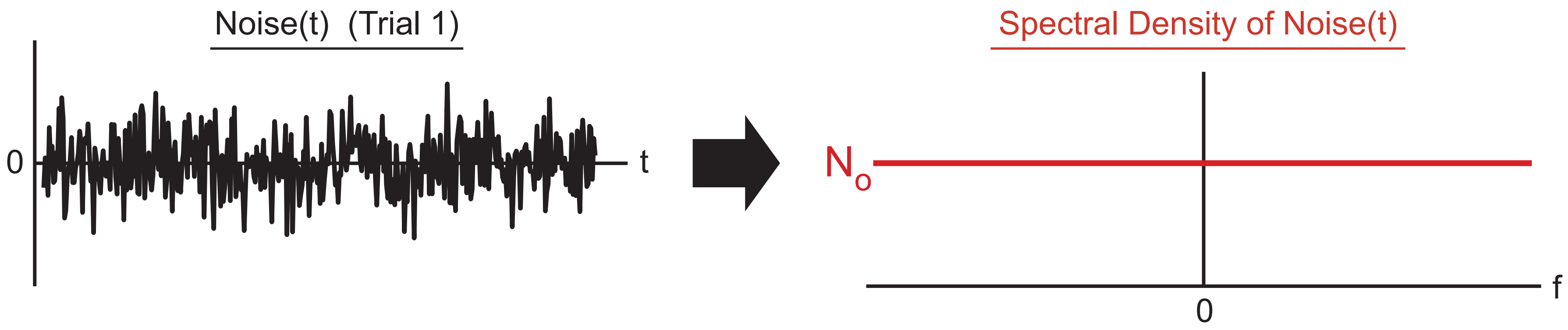
$$\delta(t) = 0 \ \text{ for } t \neq 0, \quad \int_{-\infty}^{\infty} \delta(t)dt = 1$$

  - **The power spectral density of *x(t)* is then:**

$$S_x(f) = \int_{\tau=-\infty}^{\infty} R_{xx}(\tau)e^{-j2\pi f\tau}d\tau = N_o$$

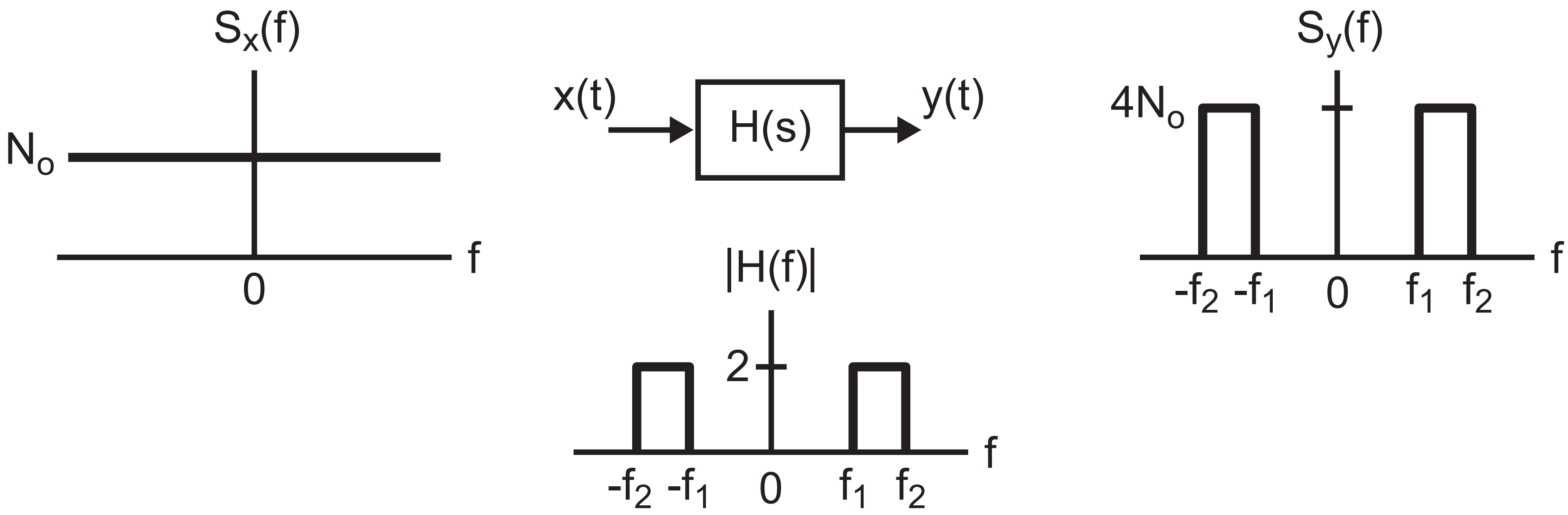
    - As with a discrete-time white process, a continuous-time white process has flat spectral density across all frequencies
    - Note that the variance of a white process is actually infinite
      - Practical “white noise” is bandlimited and has finite variance

Spectral Density of a White Process (Continuous-Time)



- **As with a discrete-time, white process, the spectral density of a continuous-time, white process is well defined**
  - **It is flat with frequency**
  - **For analog circuits, units of N_o are V^2/Hz or A^2/Hz**
  - **It is double-sided, meaning that it is defined for both positive and negative frequencies**

Spectral Density Calculations Involving Filtering

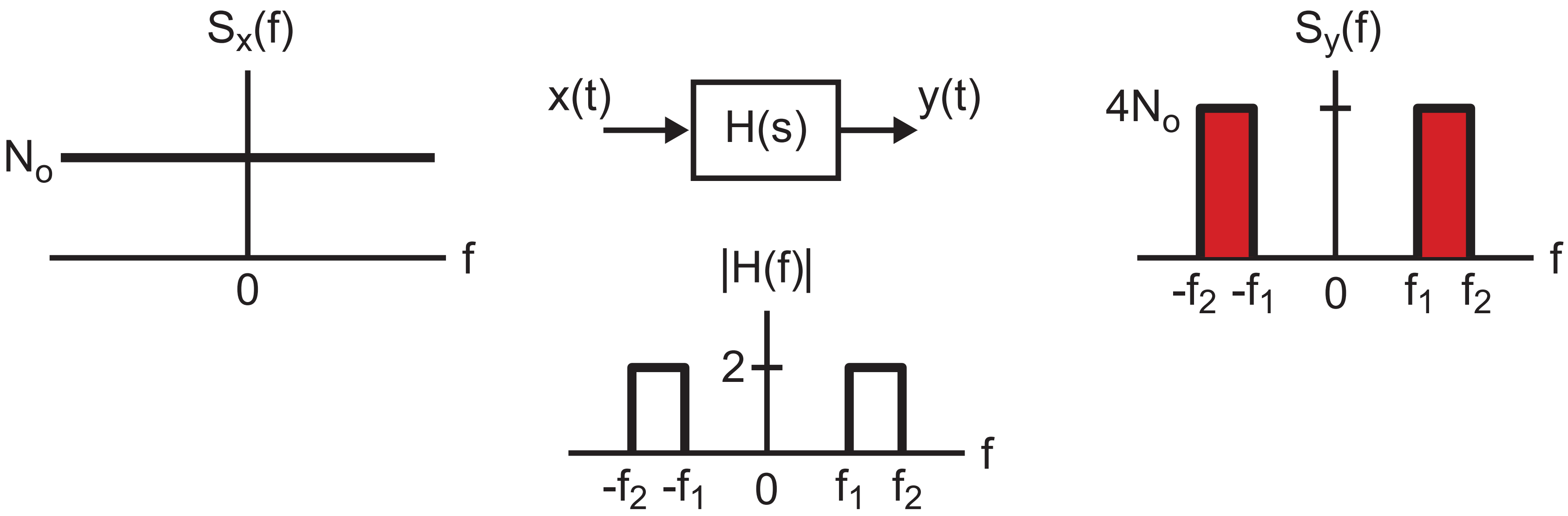


- **Assuming an input random process *x(t)* is fed into a linear, time-invariant filter *H(s)*, the resulting power spectral density of the output random process *y(t)* is calculated as:**

$$S_y(f) = |H(f)|^2 S_x(f)$$

  - **Note that filtering a white random process leads to a new random process that is no longer white**
    - The output spectral density is no longer flat across frequency
    - Different output samples in time are no longer independent

Spectral Density Calculations Involving Power



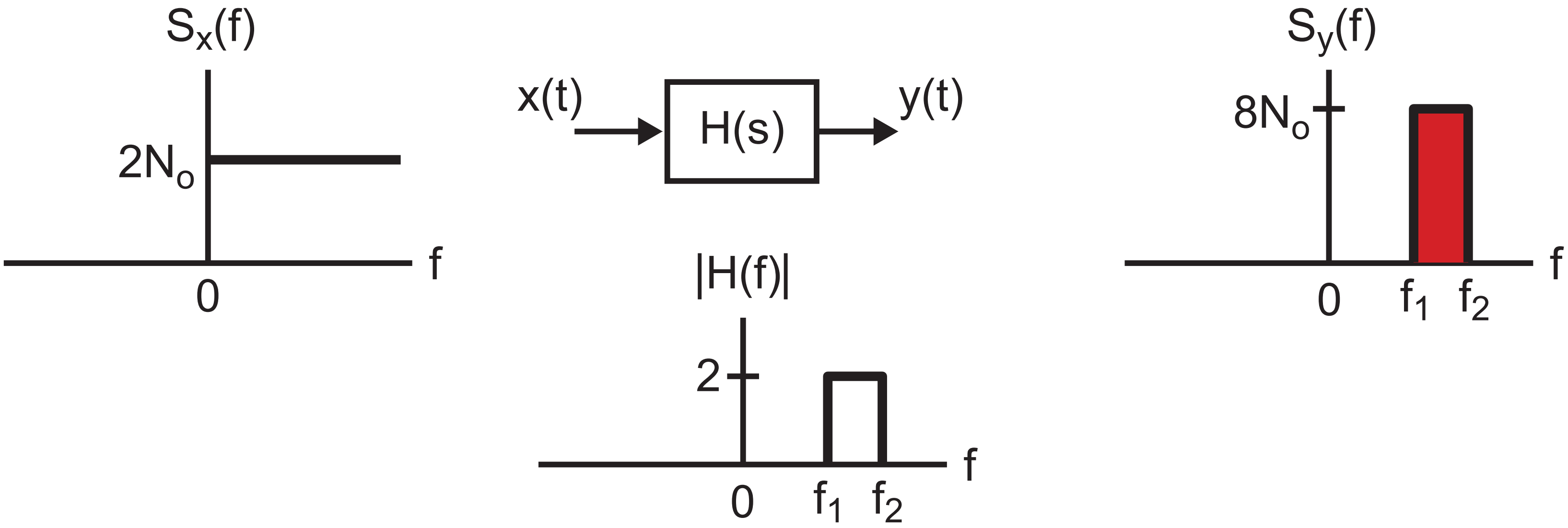
- **The power (i.e., variance) of a zero mean random process corresponds to the integration of its power spectral density**

$$P_y = \sigma_y^2 = R_{yy}(0) = \int_{-\infty}^{\infty} S_y(f)df = \int_{-f_2}^{-f_1} S_y(f) + \int_{f_1}^{f_2} S_y(f)$$

  - **Note that we can consider the power in certain frequency bands by changing the value of f_1 and f_2**
  - **In the above example:**

$$P_y = 4N_o \cdot 2(f_2 - f_1)$$

Double-Sided Versus Single-Sided Spectral Densities

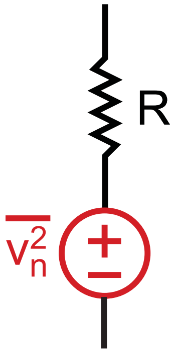


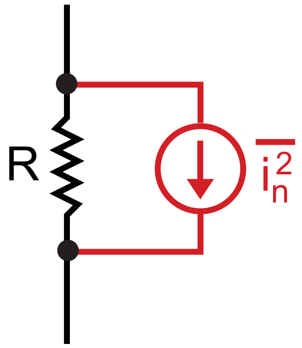
- It turns out that power spectral densities are always symmetric about positive and negative frequencies
- Single-sided spectral densities offer a short cut in which only the positive frequencies are drawn
  - In order to conserve power, the spectral density magnitude is doubled
  - For the above example: $\Rightarrow \quad P_y = 8N_o \cdot (f_2 - f_1)$

We will use only single-sided spectral densities in this class

Noise in Resistors

- **Corresponds to white noise (i.e., thermal noise) in terms of either voltage or current**





$$\overline{v_n^2} = 4kTR\Delta f \qquad \overline{i_n^2} = 4kT\frac{1}{R}\Delta f$$

  - **Circuit designers like to use the above notation in which $\overline{v_n^2}$ and $\overline{i_n^2}$ represent power in a given bandwidth *Δf* in units of Volts^2 or Amps^2, respectively**
  - ***k* is Boltzmann's constant:** $k = 1.38 \times 10^{-23} J/K$
  - ***T* is temperature (in Kelvins)**
    - Usually assume room temperature of 27 degrees Celsius

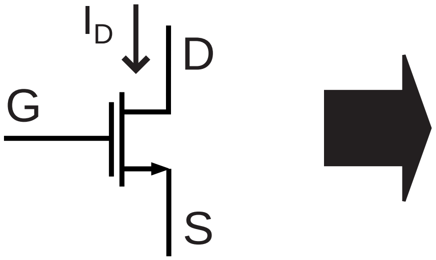
$$\Rightarrow \; T = 300K$$

Noise In Inductors and Capacitors

- **Ideal capacitors and inductors have no noise!**

- **In practice, however, they will have parasitic resistance**
  - **Induces noise**
  - **Parameterized by adding resistances in parallel/series with inductor/capacitor**
    - Include parasitic resistor noise sources

Noise in CMOS Transistors (Assumed in Saturation)



Transistor Noise Sources

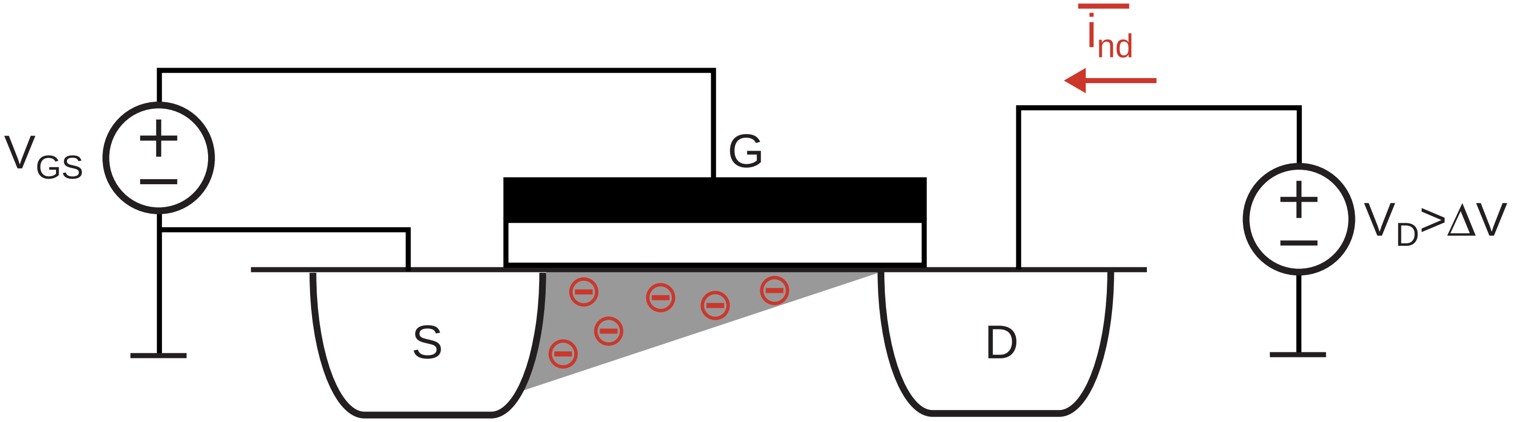
Drain Noise (Thermal and 1/f)

Gate Noise (Induced and Routing Parasitic)

- **Modeling of noise in transistors includes several noise sources**
  - **Drain noise**
    - Thermal and 1/f – influenced by transistor size and bias
  - **Gate noise**
    - Induced from channel – influenced by transistor size and bias
    - Caused by routing resistance to gate (including resistance of polysilicon gate)
      - Can be made negligible with proper layout such as fingering of devices

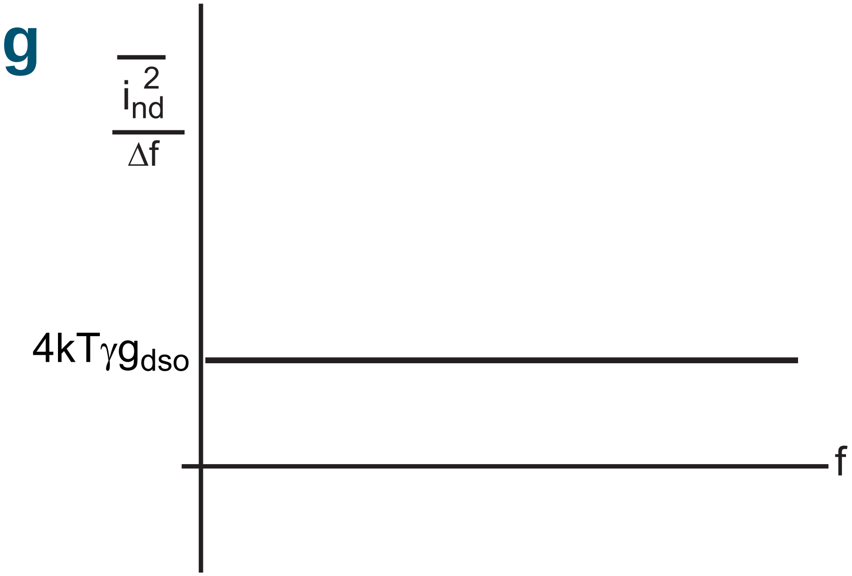
We will ignore gate noise in this class

Drain Noise – Thermal (Assume Device in Saturation)



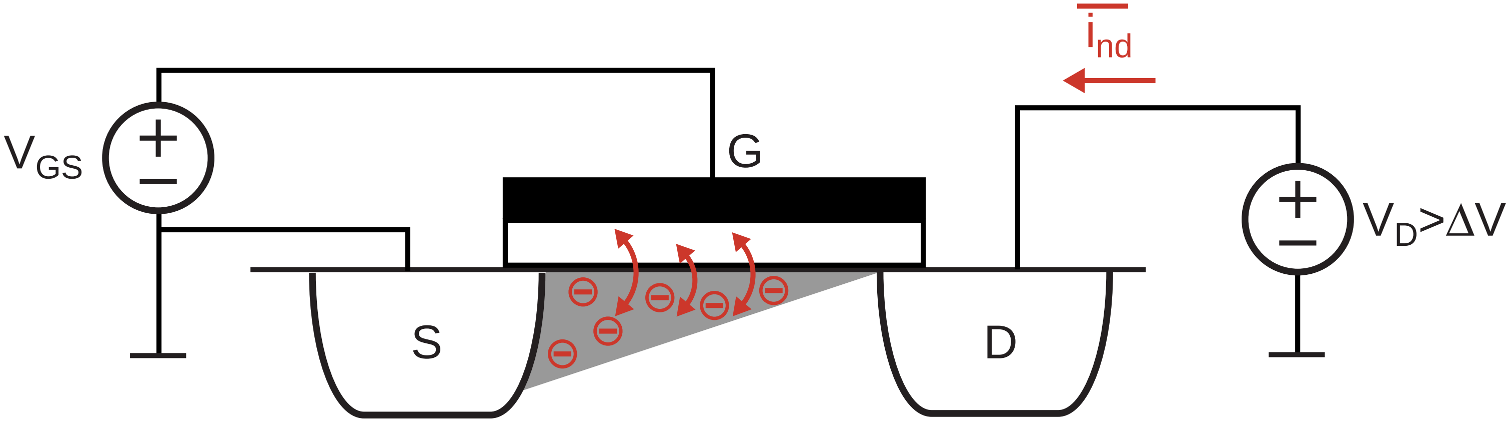
- **Thermally agitated carriers in the channel cause a randomly varying current**

$$\left.\overline{i_{nd}^2}\right|_{th} = 4kT\gamma g_{dso}\Delta f$$



  - **γ is called excess noise factor**
    - = 2/3 in long channel
    - = 2 to 3 (or higher!) in short channel MOS devices
  - **g_{dso} will be discussed shortly** (Note: $g_{dso} = g_m/\alpha$)

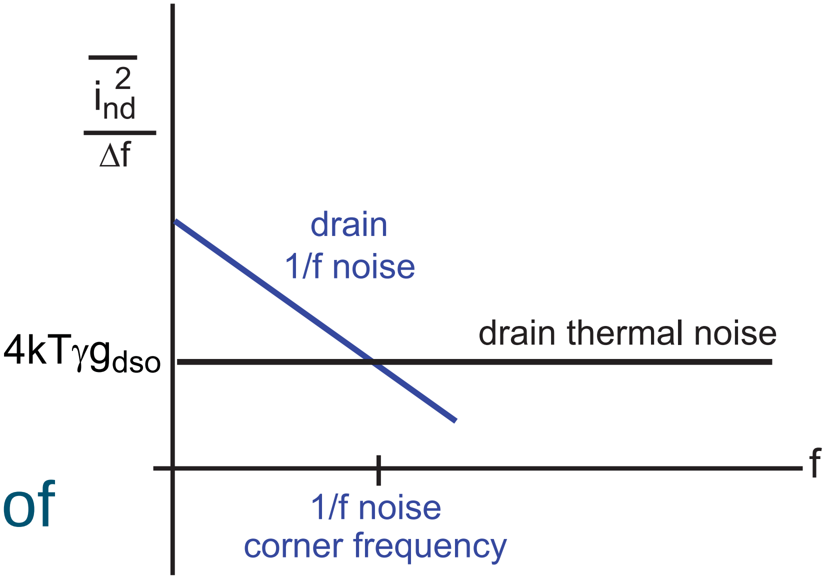
Drain Noise – 1/f (Assume Device in Saturation)



- **Traps at channel/oxide interface randomly capture/release carriers**

$$\overline{i_{nd}^2}\Big|_{1/f} \approx \frac{K_f}{f}\frac{g_m^2}{WLC_{ox}^2}\Delta f$$

  - Parameterized by *K_f*
    - *K_f* provided by fab
    - Sometimes K_f of PMOS << K_f of NMOS due to buried channel
  - **To minimize: want large area (high WL)**



Drain-Source Conductance: g_{dso}

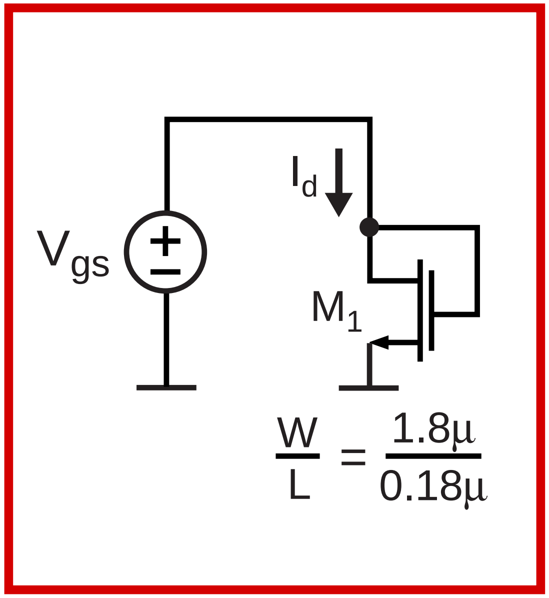
- **g_{dso} is defined as channel resistance with V_{ds}=0**
  - **Transistor in triode, so that**

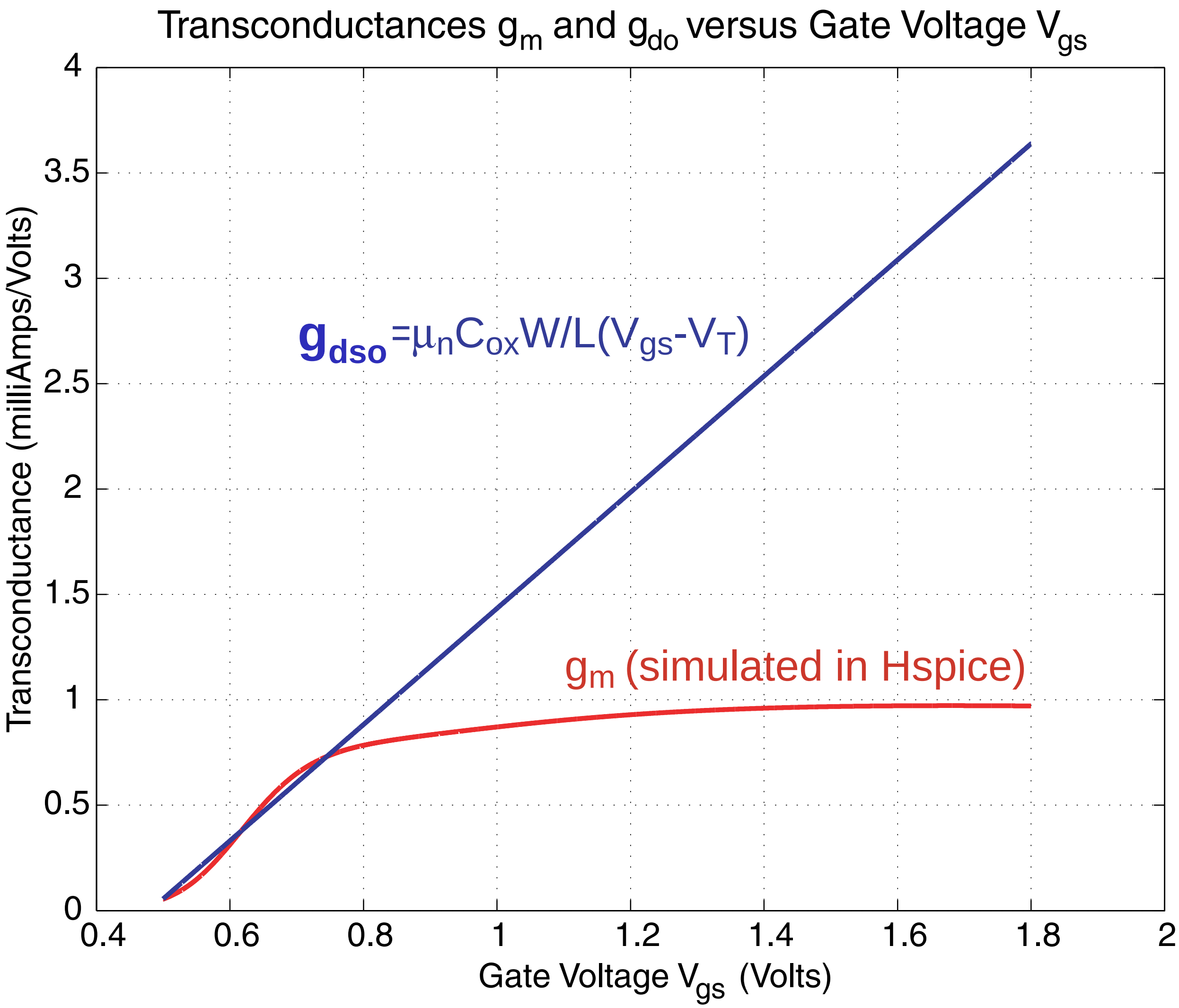
$$I_d = \mu_n C_{ox}\frac{W}{L}\left((V_{gs}-V_T)V_{ds}-\frac{V_{ds}^2}{2}\right)$$

$$\Rightarrow \boxed{g_{dso} = \left.\frac{dI_d}{dV_{ds}}\right|_{V_{ds}=0} = \mu_n C_{ox}\frac{W}{L}(V_{gs}-V_T)}$$

  - **Ideally equals g_m, but effects such as velocity saturation can cause g_{dso} to be different than g_m**

Plot of g_m and g_{dso} versus V_{gs} for 0.18μ NMOS Device

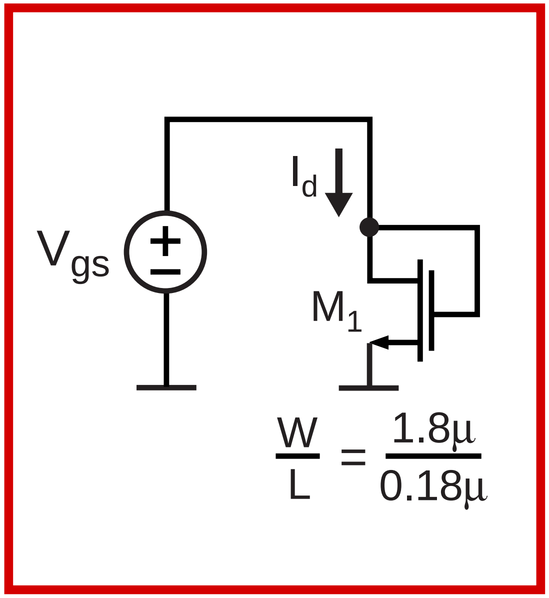


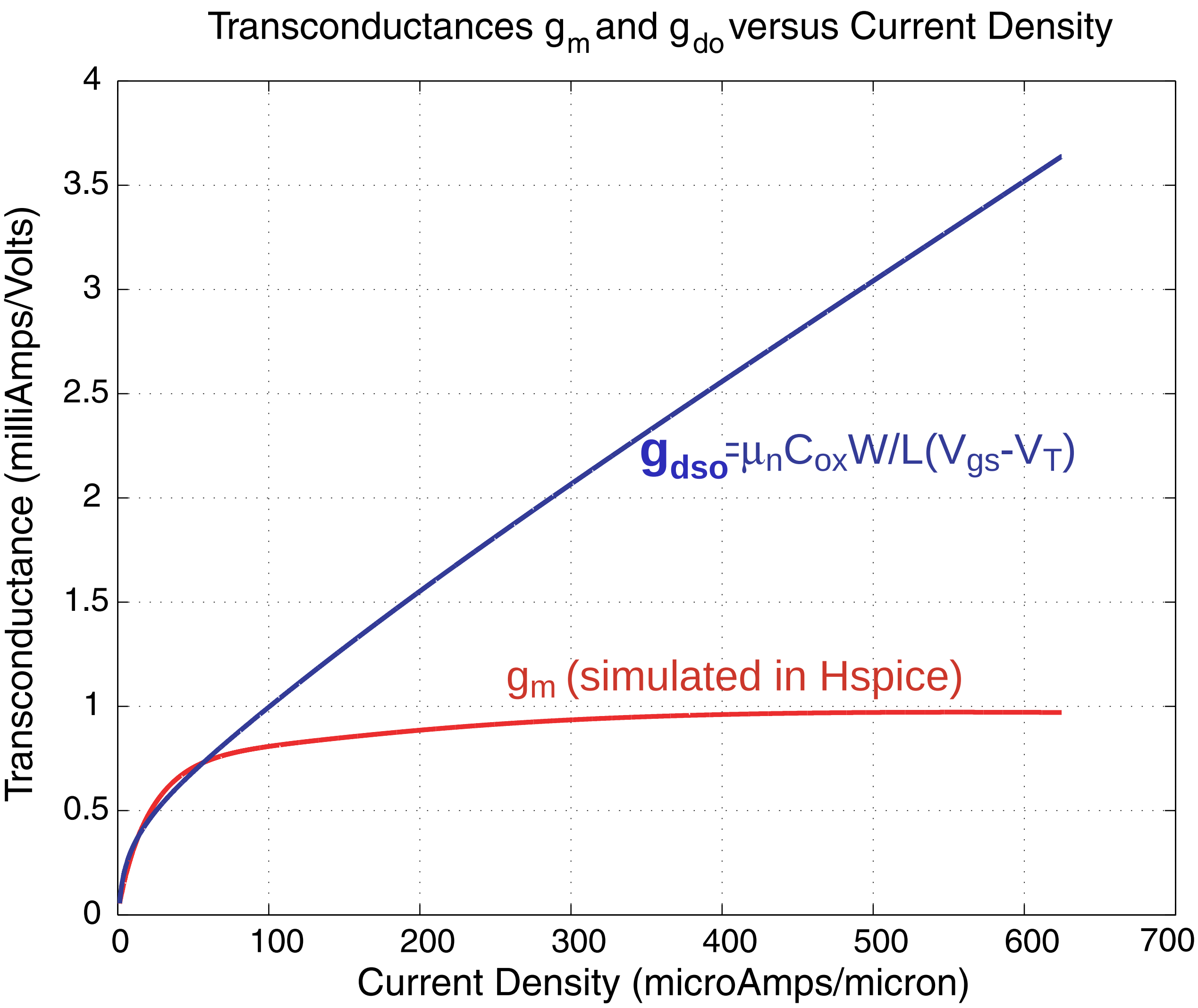


Transconductances g_m and g_{do} versus Gate Voltage V_{gs}

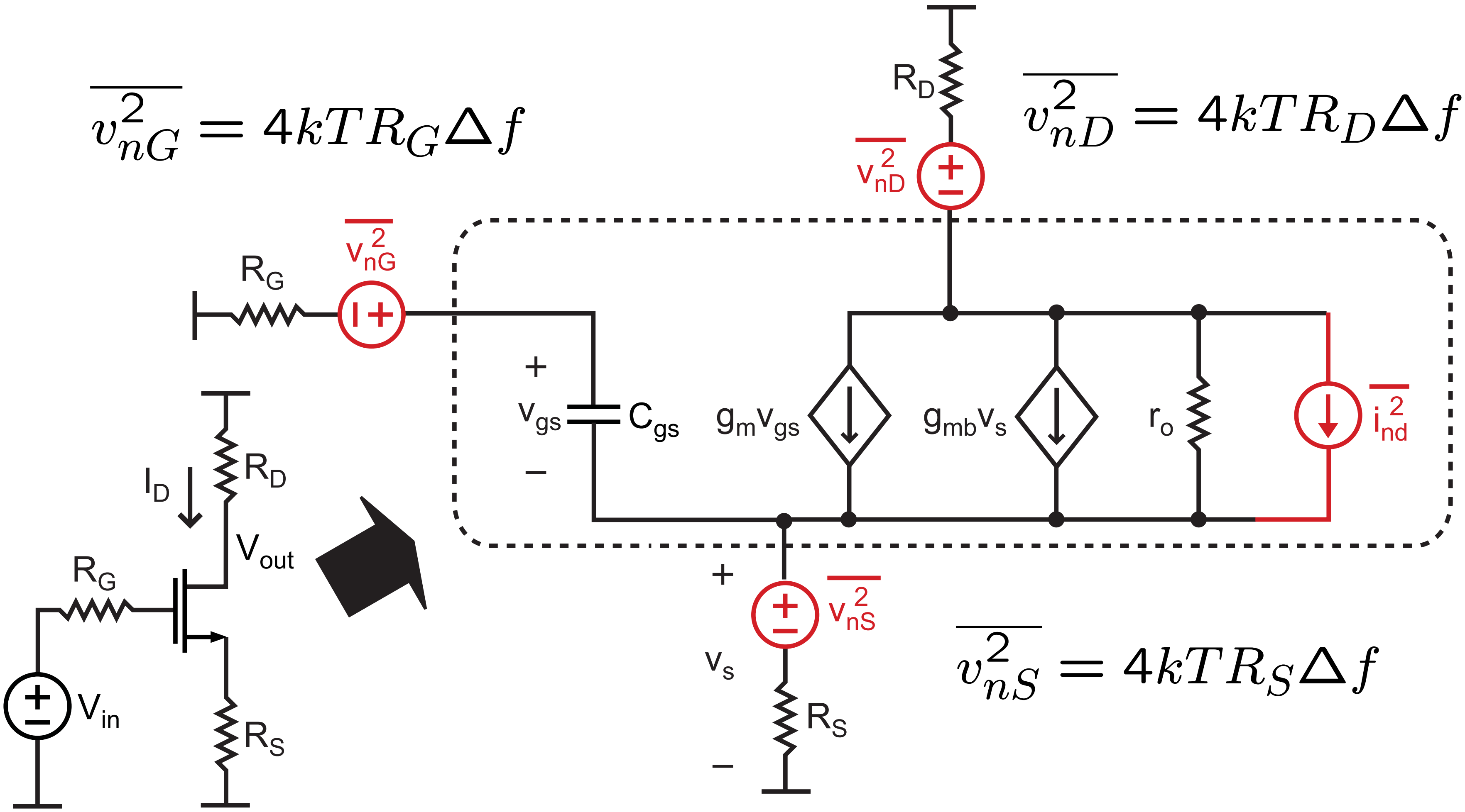
- **For V_{gs} bias voltages around 1.2 V:** $\alpha = \frac{g_m}{g_{dso}} \approx \frac{1}{2}$

Plot of g_m and g_{dso} versus I_{dens} for 0.18μ NMOS Device





Key Noise Sources for Noise Analysis



$$\overline{v_{nG}^2} = 4kTR_G\Delta f$$

$$\overline{v_{nD}^2} = 4kTR_D\Delta f$$

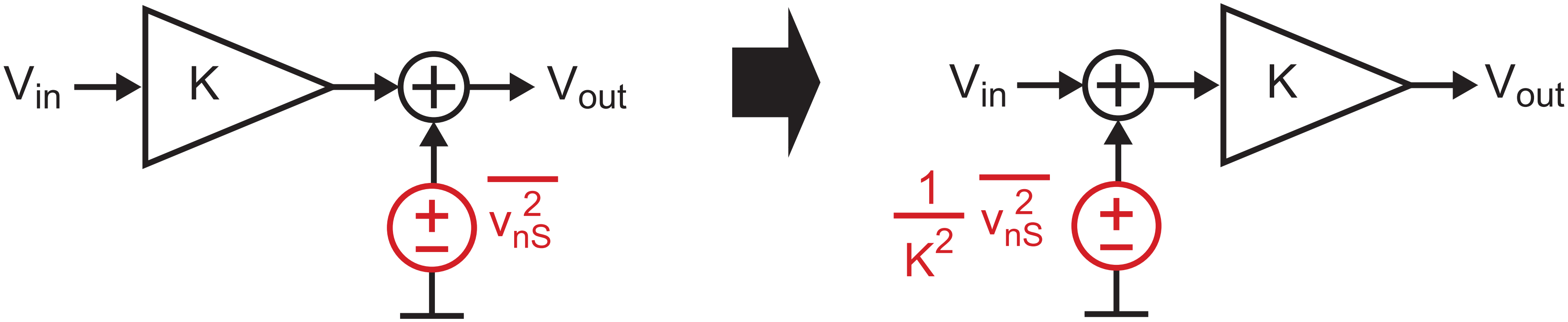
$$\overline{v_{nS}^2} = 4kTR_S\Delta f$$

- **Transistor drain noise:** $\overline{i_{nd}^2} = \underbrace{4kT\gamma g_{dso}\Delta f}_{\textbf{Thermal noise}} + \underbrace{\frac{K_f}{f}\frac{g_m^2}{WLC_{ox}^2}\Delta f}_{\textbf{1/f noise}}$

Useful References on MOSFET Noise

- B. Wang et. al., “MOSFET Thermal Noise Modeling for Analog Integrated Circuits”, JSSC, July 1994
- Jung-Suk Goo, “High Frequency Noise in CMOS Low Noise Amplifiers”, PhD Thesis, Stanford University, August 2001
  - http://www-tcad.stanford.edu/tcad/pubs/theses/goo.pdf
- Jung-Suk Goo et. al., “The Equivalence of van der Ziel and BSIM4 Models in Modeling the Induced Gate Noise of MOSFETS”, IEDM 2000, 35.2.1-35.2.4
- Todd Sepke, “Investigation of Noise Sources in Scaled CMOS Field-Effect Transistors”, MS Thesis, MIT, June 2002
  - http://www-mtl.mit.edu/wpmu/sodini/theses/

Input Referral of Noise



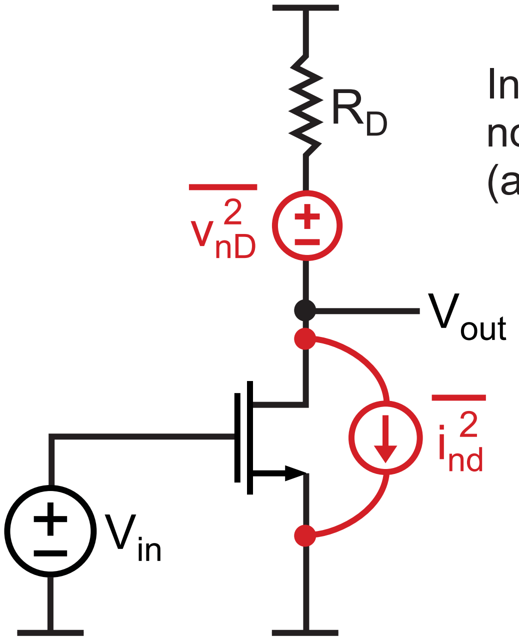
- **It is often convenient to input refer the impact of noise when performing noise analysis in circuits**
  - **To justify the above, recall that filtering a random process *x(t)* leads to an output random process *y(t)* such that**

$$S_y(f) = |H(f)|^2 \, S_x(f)$$

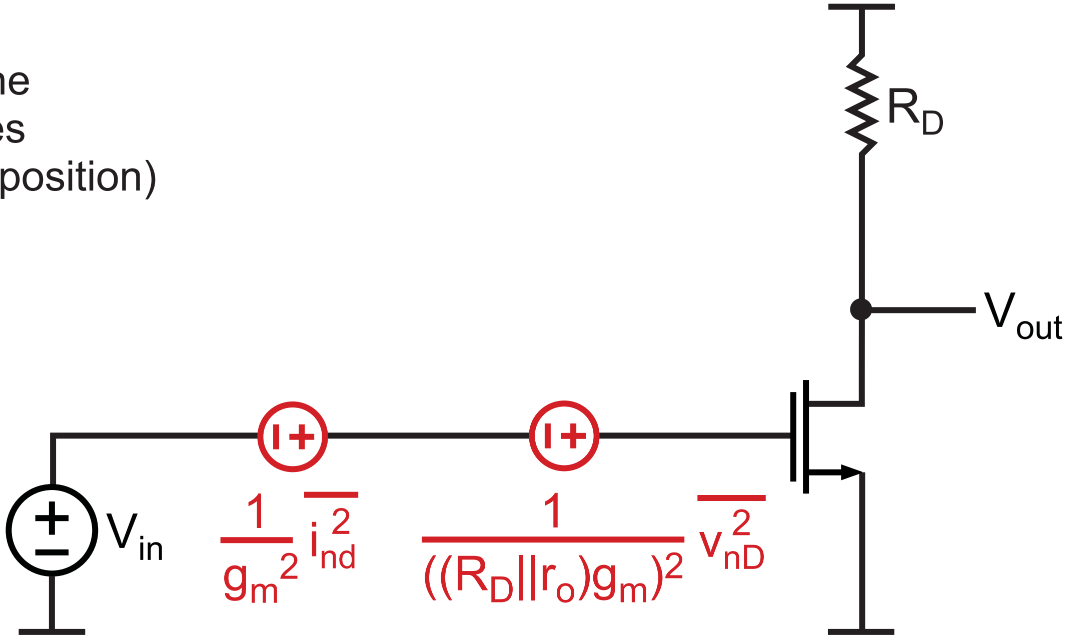
    - For the case where *H(f)* = *K* (i.e., a simple gain factor):

$$\Rightarrow \quad S_y(f) = |K|^2 S_x(f) \quad \Rightarrow \quad S_x(f) = \frac{1}{|K|^2} S_y(f)$$

Example: Common Source Amplifier

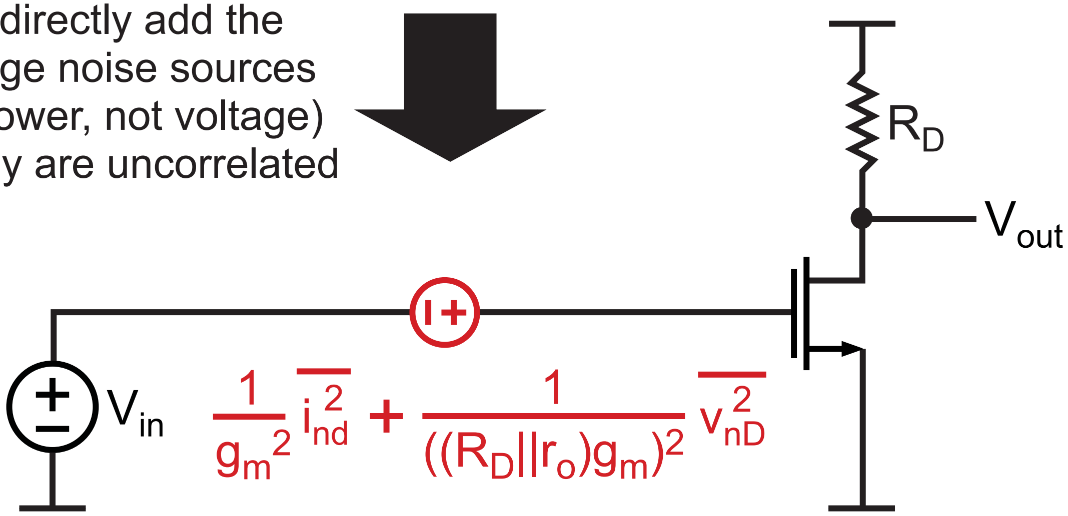


Input-refer the
noise sources
(apply superposition)



$\frac{1}{g_m^2}\overline{i_{nd}^2}$ $\frac{1}{((R_D||r_o)g_m)^2}\overline{v_{nD}^2}$

Can directly add the
voltage noise sources
(in power, not voltage)
if they are uncorrelated



$$\frac{1}{g_m^2}\overline{i_{nd}^2} + \frac{1}{((R_D||r_o)g_m)^2}\overline{v_{nD}^2}$$

- **Note that we will always assume that different circuit elements produce *uncorrelated* noise**

Summary

- Power spectral density provides a rigorous approach to describing the frequency domain behavior of the ensemble behavior of stationary, ergodic (zero mean) random processes
  - Key concepts: Expectation, Autocorrelation, Fourier Transform, Correlation, Filtering
- Circuit designers like the following "notation"
  - Single-sided rather than double-sided spectra
  - Voltage and current noise power denoted as $\overline{v_n^2}$ and $\overline{i_n^2}$
- Key noise properties of circuit elements
  - Resistor: thermal noise (white noise)
  - MOS transistor: thermal + 1/f noise
- Useful analysis tool: input referral of noise sources
  - Assumption of uncorrelated noise from different elements allows their power (i.e., variance) to be added